Index Theorems

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1 Introduction

One of the methods we have learnt in the course is Jacobian linearisation of the state-space and study the Eigenvalues around the equilibrium points to understand its nature. Such an approach is local and gives the properties of the system around the localised equilibrium point. Index theorems do not provide us with the nature of the equilibrium points, but gives us a qualitative way to understand the vector field, i.e., to know if there is a possibility of a closed orbit or to know what type of fixed points are possible within a curve etc. Important properties and definitions in index theory are given in the next section.

2 Definitions, properties and theorems

Definition: Let $\boldsymbol{C}$ be a simple closed curve which does not pass through an equilibrium point. It is called simple as it does not have any self-intersections. Such a curve is not necessarily a trajectory of the system; it is merely a hypothetical element we place in the systems vector field to investigate the nature of the field. At each point, $\boldsymbol{x}$ on the curve, consider the vector field makes an angle,

$$\dot{\boldsymbol{x}} = \{\dot{x}, \dot{y}\},$$

where $\{x, y\}$ is a point on the curve $\boldsymbol{C}$. The angle made by the vector field at a point is given as,

$$\phi = \arctan(\frac{\dot{y}}{\dot{x}}). \tag{1}$$

As we move along the curve $\boldsymbol{C}$ in a counterclockwise direction, since out vector field is assumed to be continuous, ϕ also changes continuously, and as we reach the starting point, ϕ restores to the initial value. Hence the change can only be an integer multiple of 2π. Then, the index of the curve $\boldsymbol{C}$ concerning the vector field $\boldsymbol{f}$ given as,

$$I_C = \frac{1}{2\pi}[\phi]_C,$$

where $[\phi]_C$ represents the change in the angle ϕ as it traverses along the curve $\boldsymbol{C}$.

Properties

1. Index of a closed trajectory in the vector field is $I_C = +1$.

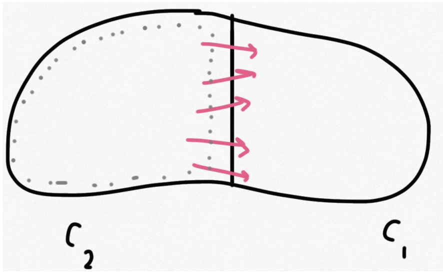


Figure 1: Sub-dividing the curve $\boldsymbol{C}$ into $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$

2. Index is additive if we sub-divide the curve $\boldsymbol{C}$ into $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$. It is because of the fact as we move on the partition, where the vector fields (pink arrows) effect while moving on $\boldsymbol{C}_1$ is cancelled by moving on $\boldsymbol{C}_2$ as shown in the Fig. 1.

3. If $\boldsymbol{C}$ is continuously deformed into $\boldsymbol{C}'$ without passing a fixed point, then the index does not change. This is because the index is an integer and it depends continuously on $\boldsymbol{C}$, which means that it must remain constant.

4. If $\boldsymbol{C}$ does not enclose any fixed point then the index, $I = 0$. This follows from the above property, as we deform the curve to make it small, then all the direction of vector fields do not change through the curve $\boldsymbol{C}$, and hence the index is 0.

5. If the sense of time is changed from $t \to -t$, then the index does not change. Hence it does not tell anything about stability.

Theorem: Any closed trajectories on $\mathbb{R}^2$ must enclose fixed points whose indices must obey,

$$\sum_{k=1}^{n} I_k = +1 \tag{2}$$

2.1 Index of a point

If only one fixed point is enclosed in a curve $\boldsymbol{C}$, then the index of the point is nothing but the index of the curve. This is because we can deform the curve $\boldsymbol{C}$ to the neighbourhood of the point and study the index around the point and this wouldn't change the index of the original curve. Indices of few points is given in Table. 1

Nature of fixed point	**Index**
Unstable node	+1
Stable node	+1
Saddle	-1
Centre	+1
Non-fixed point	0

Table 1: Indices of a few points

3 Examples

Example 1: Consider the system given in [1].

$$\dot{x} = x(3 - x - 2y) \tag{3}$$
$$\dot{y} = y(2 - x - y) \tag{4}$$

The above system has the following equilibrium points,

Point	**Nature**
$\{0, 0\}$	unstable
$\{0, 2\}$	stable node
$\{0, 2\}$	stable
$\{1, 1\}$	saddle

Table 2: Nature of equilibrium points

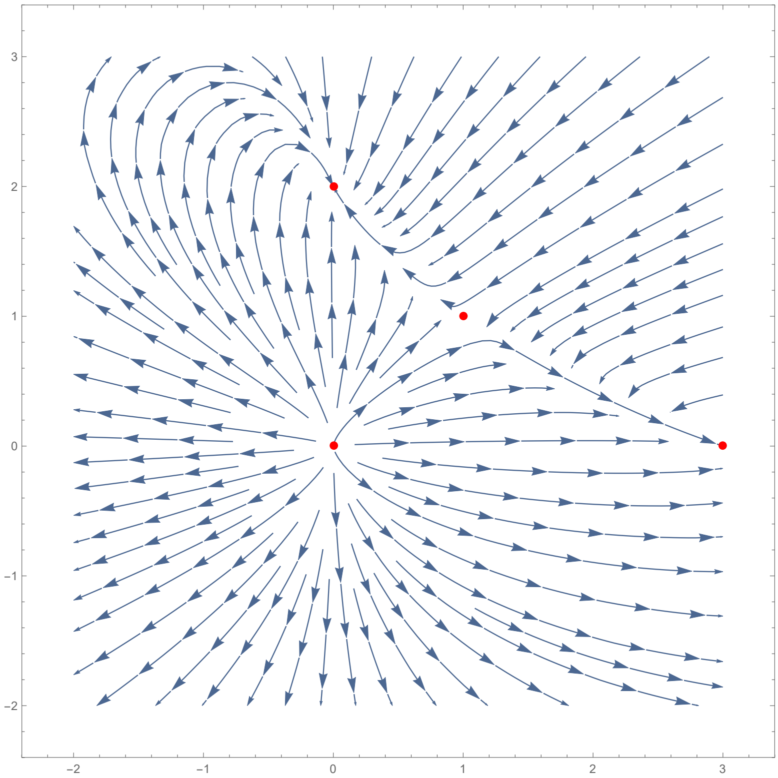


Figure 2: Phase portrait of the system 4

Using the theorem and the result Eq. 2, we can rule out the existence of closed orbits in a given system.

1. As we already know the index of a closed orbit without any fixed point in it is 0 and hence cannot have an arbitrary closed orbit.

2. Index of a curve around a saddle point is −1 hence cannot have a closed orbit around it

3. Any orbit around the stable points intersect with the existing vector field around it as the red dotted curve shown in Fig. 3.

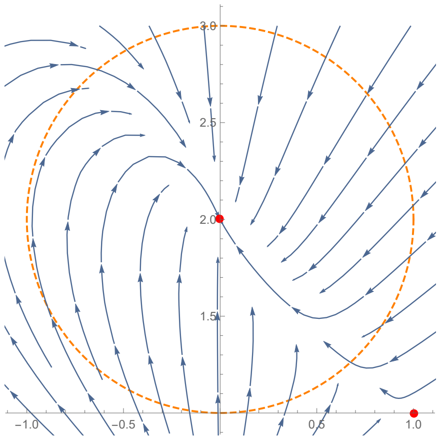


Figure 3: A closed trajectory around stable equilibrium point

From the phase portrait Fig. 2, we can see that there is are no closed orbit around any of the fixed points. All the above observations can be verified from the same. A similar method is also illustrated in [2]. The red dots indicate the fixed points of the system.

Example 2: Consider the bead and the ring problem as shown in the figure, Fig 4.

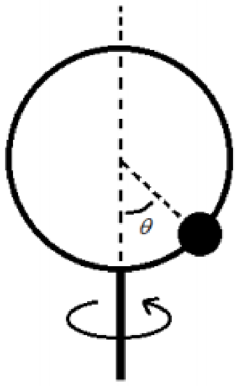


Figure 4: Bead on a rotating ring system. Image Source:Google Images

The system dynamics can be written as,

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{2}\sin(2x_1)\ \omega^2 - \frac{g}{r}\sin(x_1),\end{aligned}$$

where the states x_1 and x_2 represent the angle θ and angular velocity $\dot{\theta}$ respectively. The equilibrium points of this system are,

$$\theta = 0, \pi, \pm\arccos(\frac{g}{r\omega^2}),$$

where r is the radius of the ring, ω is the angular velocity of the ring, and g is the acceleration due to gravity. As long as the term inside the arccos remains within ± 1, we have 4 real solutions; otherwise, we have only 2 real solutions. This is a classic case of pitchfork bifurcation.

Consider the case where ω is below critical velocity for bifurcation, since π is an unstable node, there cannot be any closed loop trajectory around it as can be seen from the actual phase portrait in Fig 6a.

But we can have one around the stable equilibrium point, i.e., 0. If we consider a small circle around this equilibrium point as shown in Fig. 5. Computing the index gives us, $I_C = 1$,

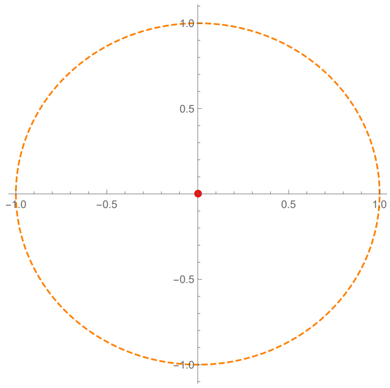
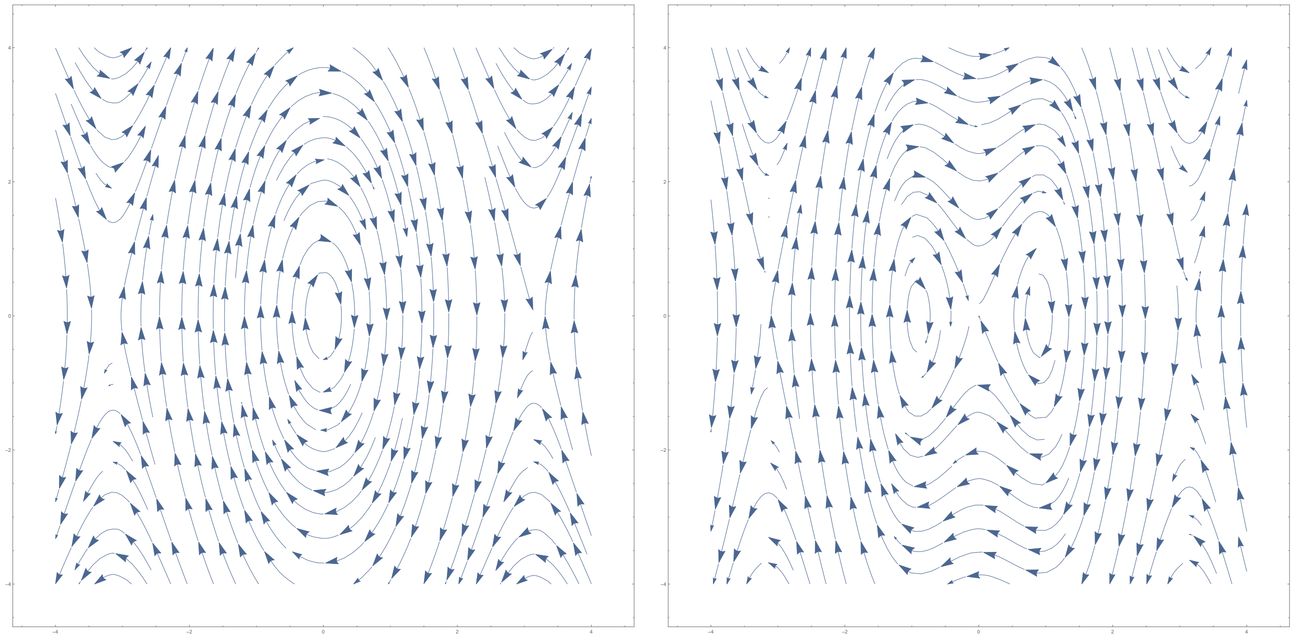


Figure 5: Selected circle (curve $\boldsymbol{C}$) around the equilibrium point



(a) Phase portrait of the bead-ring system below critical velocity i.e., before bifurcation

(b) Phase portrait of the bead-ring system above critical velocity i.e., after bifurcation

as the curve contains one stable node within it. But as we move above the critical velocity, the stable node becomes a saddle point and the index changes from $+1 \rightarrow -1$. We can further take a bigger circle around the three equilibrium points, and since we have one saddle and two stable equilibrium points, we can conclude that a closed trajectory is possible enclosing all the three equilibrium points as can be seen from the actual phase portrait in 6b.

4 Conclusion

The idea here is similar to the one we use in electrostatics, where we use Gaussian surfaces to understand the charge enclosed within the surface. We have demonstrated the use of index theorems to verify the existence of a closed loop trajectory in Example 1, and Example 2 investigated the change in the nature of the equilibrium point in a bifurcation and also found the possibility of a closed trajectory after bifurcation. Hence, index theorems are helpful in qualitatively understanding the systems phase portrait and the nature of trajectories.

References

[1] S. Strogatz, M. Friedman, A. J. Mallinckrodt, and S. McKay, "Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering," *Computers in Physics*, vol. 8, no. 5, pp. 532–532, 1994.

[2] M. Vidyasagar, *Nonlinear systems analysis.* Siam, 2002, vol. 42.